THE

MATHEMATICAL GAZETTE.

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REPORT OF THE COUNCIL, 1905.

THE Council has very little to report concerning the year 1905. It has been very quiet, and barren of events.

The number of new members elected during the year is 26, making a total membership of 395 at the present time.

Six numbers of the Mathematical Gazette have been issued during the year, and the Council desires to express its high appreciation of the labours of Mr. Greenstreet and those who have assisted him in his editorial work.

The Report on the "Teaching of Elementary Mathematics" has been reprinted during the year, with the addition of reports on the teaching of Elementary Mechanics and on advanced School Mathematics, and also with the regulations for Geometry in the Cambridge Previous Examination. In its enlarged form the pamphlet has been distributed to all old members of the association and also to those who have been elected during the year.

After the lapse of two years the Roll of Members has been reprinted, and a proof copy lies on the table. The Council hopes that the new Roll will be ready for distribution with the March number of the Gazette.

The Association had its first dinner in the evening after the close of the last general meeting. It was a very pleasant gathering at which about 25 members and guests were present. The second dinner will be held this evening. The Council feels that the Association is deeply indebted to Mr. C. S. Jackson, who has undertaken and carried out most successfully the whole of the arrangements on both occasions.

In accordance with the Rules of the Association two members of the Council retire annually. The retiring members this year

are Mr. S. Barnard and Mr. C. Godfrey. The Council nominates Mr. Roseveare of Harrow, and Mr. Whipple of Merchant Taylors' School to fill the vacancies.

REPORT OF THE COMMITTEE ON THE TEACHING OF ELEMENTARY MATHEMATICS, 1905.

THE Committee has not met during the year 1905.

In February Mr. Hawkins drew the hon. secretary's attention to the fact that the Law Society and the Surveyor's Institution still required a knowledge of Euclid in their preliminary examinations. After some correspondence the Law Society in July removed from their regulations the words "No proof can be accepted which violates Euclid's order," and substituted "Proofs other than Euclid's will be accepted and simple riders will be set." In November the Surveyor's Institution decided "to allow candidates the choice of either Euclid or geometry." During the year the Committee's reports have been collected and republished under one cover.

A. W. SIDDONS,
Hon. Sec.

AN APPEAL.

SEVERAL members of the Association expressed their conviction, at the General Meeting and at the subsequent Dinner, that there should be more articles in the *Gazette*, and more papers read at the Meetings, on the purely pedagogic side of mathematics. It is right and necessary that the general development of the various branches of our Science should not be lost sight of in an organ such as the *Gazette*, and the reader must be kept in touch by reviews with what is being done in school text-books both at home and elsewhere. But the special aims of the Association (and the object of the *Gazette* is to secure by publicity the realisation of those aims), is to *improve the teaching* of our subject in our schools. With this view the Editor is in cordial agreement. He asks the readers of the *Gazette* to come to his assistance in the matter, to propose subjects for discussion, to offer papers, to ventilate their special difficulties in a "Question and Answer" column, in short to do for the *Gazette* what no Editor can do single-handed. If this proposal is taken up with any vigour by our readers, it should be quite possible to arrange for the regular appearance in the *Gazette* of a series of Symposia, consisting of (1) articles from the pens of its members and (2) a general summing up of the *pros* and *cons* of the points at issue by an authority who combines academic distinction with teaching experience. The latter is easy to find. It is the former who as yet have not come to the front in our Association. The Council has said in the Annual Report that this last year

has been a quiet year. We are not sure that this may not be meant for a piece of biting criticism, suggesting that though more remains to be done we have not done it. Be that as it may, we appeal to the readers of the *Gazette* for their cordial co-operation in providing the material for such discussions as are indicated above. Apart from our differences as to detail in the processes of teaching, there are many other topics which should be of interest to the majority as well as to special groups of our members. Those who are engaged in Preparatory Schools should be glad to expound their views of what is feasible with boys of the age with which they deal, and the Public School Teachers of Mathematics should be glad to indicate the lines along which training should proceed, and any way of improving the preparation of the boys who are ultimately to come under their care. The fairness of the papers set at the various examinations, and their strict adherence to the official syllabus may also prove a fruitful subject of discussion.

At the moment of writing we have received a letter from a prominent member of the Association and of the Council, expressing the opinion that all teachers of Mathematics are at present in a state of difficulty, more or less. The text-books (in Geometry, especially) seem to him to incline too much to practical in preference to theoretical work, and he hazards the conjecture that the majority of good teachers regard this as an evil. We imagine that the complete change that has taken place in some of the examinations will be found by many a more real "difficulty" than even the text-books, in so far as the sound teacher is independent of the text-book. There is one syllabus which is reported to inform candidates that "systematic knowledge is not expected"(!) What kind of knowledge is expected is not stated. Subjects are increased out of all bounds, and each is intended to be a mere conglomeration of scraps. Nearly all the questions are drawing or calculations. In one case the only theoretical question was taken from Book XI., and it could not be done without the aid of Book VI. Again, it is only necessary to remind those who prepare boys for Mathematical Scholarships of a case which, if "gibbeted" at the time, would have led to greater care in the future conduct of such examinations. In the questions set in the morning and afternoon papers at one of the Colleges no less than four or five gave incorrect results. Also, the way in which some examiners habitually stray from the lines of the syllabus is ever the cause of discontent among teachers.

We feel it is only necessary to call the attention of our readers to some of the ways in which the *Gazette* may be brought more into touch with every side of their daily labours to secure the suggestions for which we appeal. There is no lack of initiative

to be found among the members of the various associations founded to promote improvement in the teaching of other subjects of the curriculum, though with one exception they have not the advantage of an organ for the promulgation of their opinions. And, having such an organ, it seems a pity, to say the least of it, that our members do not utilise it to the utmost for the purposes which are the primary objects of the Association.

THE EDITOR.

SOME NOTES ON CERTAIN THEOREMS IN HIGHER TRIGONOMETRY.

I ADMIRE the ingenuity of the method by which Prof. Nanson establishes the expansions of $\sin x$ and $\cos x$ as power series in x. But I must confess that his attempt to 'simplify' the 'accepted accurate' proofs of these expansions seems to me a fundamentally mistaken one. Nor can I admit that the result is at all satisfactory 'from the elementary didactic point of view.' His proof is to my mind essentially an artificial *verification*, and altogether *unnatural*, in the sense that it has no place in any natural and logical way of developing the ideas which lead to the result. I speak with diffidence as one who has had far less experience of teaching than Prof. Nanson. But I am convinced that my criticism is a fair one: and I think that many of the proofs usually given, in English text-books, of many of the theorems of 'Higher Trigonometry,' are open to fair criticism on similar grounds, when (as is too seldom the case) they are not to be condemned for 'hopeless complexity' or utter lack of force.

And if this is so, so it will and must be so long as this subject continues to occupy its present place in the mathematical course. At present, if one may judge by results, boys are taught these theorems quite prematurely and before they can have had the opportunity of acquiring anything like a sufficient grounding of general mathematical knowledge. The great majority of boys (I am speaking, of course, only of those whose abilities are up to something approaching a scholarship standard) never really grasp the meaning of them, and one of the most fascinating and instructive regions of mathematics is, from an educational point of view, practically wasted.

How are we to define what we mean by 'Higher Trigonometry'? It is, I suppose, the theory of the exponential, logarithmic, and circular functions, and of the series and products associated with them, for real and complex values of the variable. It is true that a certain part of this theory is usually lopped off and labelled as a part of 'Higher Algebra,' but the most conservatively-minded of teachers would probably admit the

absurdity of this. At any rate this is what I mean by 'Higher Trigonometry,' and there seem to me to be four theorems or groups of theorems which may be said roughly to mark the stages of our progress in the subject. These are:—

(A) De Moivre's theorem for a real and rational index.

(B) The exponential theorem, that is to say the theorem that if x is real and rational one of the values of $\left(1+\frac{1}{1!}+\frac{1}{2!}+\dots\right)^x$ is $1+\frac{x}{1!}+\frac{x^2}{2!}+\dots$, or that one of the values of e^x is exp. x.

(C) The theorem that if x is real exp. $(ix)=\cos x+i\sin x$.

(D) The factor-theorem for $\sin x$.

It is with theorem (C) that I am particularly concerned at present. But there are one or two remarks which I should like to make about the other theorems.

(A), I think, presents no difficulty, as soon as the elements of the Argand diagram are mastered. *Irrational* powers should at this stage be severely and *explicitly* left alone. Any further extensions of the theorem should in any case be regarded as corollaries of (C); and so indeed *may* (A) itself; but the more direct treatment is better put first. (B) is a direct corollary from the multiplication theorem for exp. x. This last theorem should of course be proved for *all* values of x: it naturally requires some elementary notions with regard to convergence, and how far these should be insisted upon is a matter for judgment in particular cases. The theorem

$$\lim_{n=\infty}\left(1+\frac{x}{n}\right)^n=\exp.x$$

(with whatever degree of rigour or generality it may be stated) is instructive and important, but essentially accessory and not fundamental, and to base the exponential theorem on it, although possible, is logically quite wrong. The logarithmic expansion is more difficult and may well be postponed a while. Finally (D) is *really* difficult, and the question of what to do about it is a real *crux* in the teaching of trigonometry. There is no direct and rigid proof which is at all possible for boys, and I cannot think that anyone who has had to give a proof in lectures on the theory of functions will imagine that any indirect method of procedure is likely to lead to better results. One has only to read the account of the matter in Harkness' and Morley's excellent *Introduction to the Theory of Analytic Functions* to understand that we have here to deal with difficulties which we cannot evade. Fortunately a rigid proof is in no way necessary for profitable employment of the result, and the result is easily made *plausible* from either point of view or both.

So much for (A), (B), and (D). With regard to (C) there is much more to be said, as the theorem may be approached from so many points of view.

(i) In the first place we may postpone the theorem until a knowledge of Taylor's theorem has been acquired. There is then no difficulty at all. But although I am heartily in favour of pushing the Calculus, both Differential and Integral, as far forward in mathematical teaching as can possibly be done, I hardly think this theorem need wait so long, and we certainly lose something if it does. Moreover, if we adopt this method, the theorem (C) will naturally appear as a *consequence* of the expansion-theorems for $\sin x$ and $\cos x$. This, to my mind, is an inversion of the natural logical order, and in so far an argument against this method of procedure.*

(ii) The same argument also tells against the second possible method, which is that adopted in Chrystal's *Algebra*. The principle of this method is to give an accurate proof, independent of the notions of the Calculus, of the expansions, and to deduce (C) from them. Every deference is due to so eminent an authority as Prof. Chrystal; but I cannot help thinking that, from the elementary 'didactic' point of view, this method combines every possible disadvantage.

(iii) Thirdly, we may presuppose a little more knowledge of the foundations and a little less of the superstructure of the Calculus, and we may serve up the old line of argument,

$$y = \cos x + i \sin x, \qquad \frac{dy}{dx} = -\sin x + i \cos x;$$

$$\frac{dy}{y} = i dx,$$

which may be made to lead to the desired result in a perfectly simple and satisfactory way. This is in substance the way followed by Jordan in his *Cours d'Analyse*, and is the way I should generally choose myself. The only point about it is that it essentially involves the ideas of y varying along a circle in the Argand diagram in such a way that dy/y is purely imaginary, and of the logarithm as a many-valued inverse of the exponential function of a complex variable.† These ideas are simple and fundamental, and should be acquired much earlier than is usually the case; and there is nothing essentially new, to anyone who understands the meaning of an integral, in the notion of $\int \frac{dy}{y}$ when y varies along any contour. And I am sure that anyone

* A modification of this method has, I find, been suggested by Prof. Bromwich (*Math. Gaz.*, vol. iii., p. 85). There is a good deal to be said for the line of proof which he adopts, but I cannot regard it as the best.

† It has to be assumed that the exponential series can be differentiated term by term. I should never scruple, at this stage, to make assumptions such as this.

who has not yet acquired these ideas will derive more profit from the attempt to do so than from puzzling himself with theorems which are in reality applications of them: for this is the real *genesis* of (C).

(iv) Fourthly, we may follow Stolz and Gmeiner's method in their *Einleitung in die Funktionentheorie*. They presuppose less than is necessary if the preceding method is adopted.

Suppose $$\exp.(ix)=u+iv.$$

Then $$u^2+v^2=\exp.(ix).\exp.(-ix)=1,$$

and the modulus of $u+iv$ is unity. Hence

$$u+iv=\cos\phi+i\sin\phi,$$

where ϕ is *some* real angle: we may suppose $0\leqq\phi<2\pi$. In particular

$$\exp.(i)=\cos\phi+i\sin\phi,$$

where $0<\phi<2\pi$, as $\exp.(i)$ is evidently not equal to unity. By De Moivre's theorem

$$\exp.(\xi i)=\cos\xi(\phi+2k\pi)+i\sin\xi(\phi+2k\pi),$$

where k is an integer. If $\phi+2k\pi=\psi$, and we equate imaginary parts, we obtain

$$\xi-\frac{\xi^3}{3!}+\ldots=\sin\xi\psi,$$

and since $$\lim_{\xi=0}\frac{\sin\xi\psi}{\xi}=\psi$$

it follows that $\psi=1$.

This method, which I had not seen before the publication of Stolz and Gmeiner's book, is very simple and there is a great deal to be said in its favour.

(v) Fifthly, we may establish the two equations

$$\lim_{n=\infty}\left(1+\frac{x+iy}{n}\right)^n=\exp.(x+iy),$$

$$\lim_{n=\infty}\left(1+\frac{x+iy}{n}\right)^n=e^x(\cos y+i\sin y),$$

directly and independently. This method (in conjunction with (ii)) is followed in Hobson's *Trigonometry*. As I have indicated already, I cannot consider it a good one, and it is probably more difficult than any of the others.

It will be observed that (iii) (iv) and (v) agree with one another in making the expansion-theorems for $\cos x$ and $\sin x$ corollaries of (C), whereas in (i) and (ii) this interdependence is inverted. For this reason the three latter methods appear to me to be logically preferable, and for my part I should discard (ii) and (v) and adopt (iii) or (iv), and if possible, *both*, though of course not losing sight of (i) for such pupils as are familiar with Taylor's theorem. The simplest method is doubtless (iv),

but (iii) seems to me by far the most instructive. And if the objection is raised that the range of ideas required for (iii) is impossible for schoolboys, I should reply first, that, if this is so, then Higher Trigonometry, except in so far as it is concerned with *finite* series and products only, had better be postponed; and secondly, that I do not believe that it is so, and that a wider range of general ideas and less practice in detailed application of them is precisely what is wanted in elementary teaching, in Analysis even more than in Geometry. G. H. HARDY.

A NEW CUBIC CONNECTED WITH THE TRIANGLE.

THE locus of the points of contact of tangents from A to the conics of the confocal system with foci B, C is the same curve as the locus of the feet of normals from A to conics of the system. For any tangent AP to one of the conics is a normal AP to the orthogonal conic of the system through P. Hence if the co-ordinates of A referred to BC and the perpendicular bisector of BC be $x_0 y_0$ the equation of this locus is the eliminant of λ between

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1, \quad \frac{x_0-x}{\frac{x}{a^2+\lambda}}=\frac{y_0-y}{\frac{y}{b^2+\lambda}}.$$

Thus the locus is the cubic

$$[x(x-x_0)+y(y-y_0)][yx_0-xy_0]=(a^2-b^2)(x-x_0)(y-y_0).$$

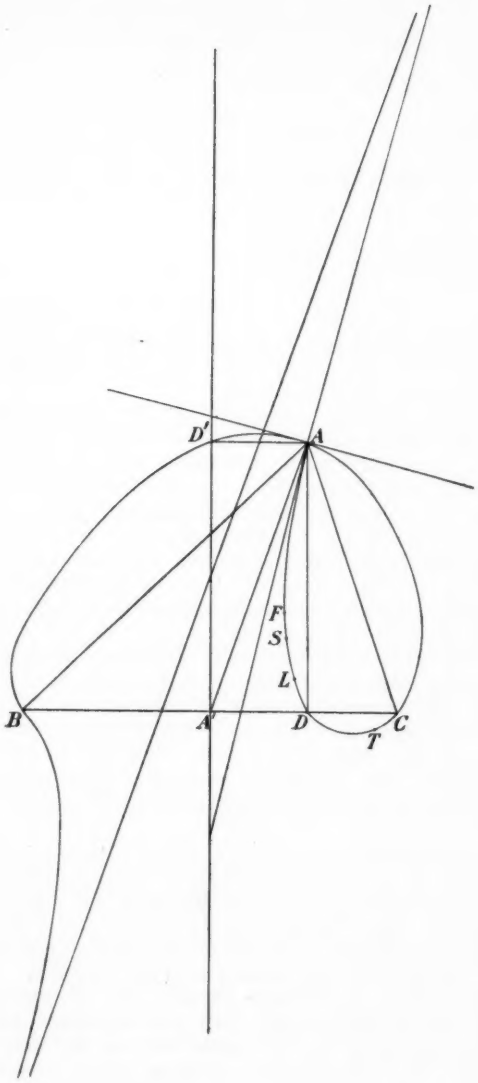
The cubic is shown in the accompanying diagram.

The points $(\pm\sqrt{a^2-b^2},\ 0)$ and $(0,\ \pm\sqrt{b^2-a^2})$ are on it, *i.e.* it passes through the real pair of foci BC of the system and the imaginary pair. It also passes through the circular points at infinity.

The points $(x_0, 0)$, $(0, y_0)$ are on it. Thus it passes through the feet of the perpendiculars from A on BC and the perpendicular bisector of BC. This can be seen geometrically since these feet DD' are the feet of normals from A to limiting conics of the system.

The point A is on it and is a node the tangents at which are the bisectors of the angle A, as is seen by considering the limiting ellipse and hyperbola through A.

Again, if P is any point on the locus, AP being a tangent to one of the conics bisects the angle BPC either internally or externally and conversely, any point having this property is on the locus. But Fermat's point, F, has this property, hence it is on the cubic. Also the focus of the parabola which touches AB, AC at B and C has this property. Hence this focus S is on the cubic.


D'
A
F
S
L
B
A'
D
T
C

The real asymptote of the cubic is

$$yx_0 - xy_0 = \text{Lt.}\ \frac{(a^2-b^2)(x-x_0)(y-y_0)}{x(x-x_0)+y(y-y_0)},$$

where $x=\frac{x_0}{t}$, $y=\frac{y_0}{t}$ and t approaches ∞.

Thus it is $$yx_0 - xy_0 = \frac{(a^2-b^2)x_0y_0}{x_0^2+y_0^2}.$$

Thus it is parallel to the median AA'. The intercepts on the axes are

$$-\frac{(a^2-b^2)x_0}{x_0^2+y_0^2},\quad \frac{(a^2-b^2)y_0}{x_0^2+y_0^2},$$

$$i.e.\ -\frac{A'C^2}{AA'^2}.A'D,\quad \frac{A'C^2}{AA'^2}.AD.$$

Any conic of the system is cut by the cubic in 6 points which are the feet of the four normals and the points of contact of the two tangents from A to that conic.

The external portions $AD'B$, AC of the cubic consist of the points of contact of tangents from A to those ellipses of the system for which these are real, starting with the ellipse through A and ending with the line BC. The portions $A\infty B$ and $AFDC$ consist of the points of contact of tangents from A to those hyperbolas of the system for which these are real, starting with the hyperbola through A and ending with the line hyperbola, vertices BC. Of these tangents the pair corresponding to any one hyperbola touch the same branch until one of them becomes parallel to the asymptote of the cubic, *i.e.* coincides with AA', after which they touch different branches. Now the second tangent of the pair, one of which is AA', is the isogonal line to AA', *i.e.* is AK, where K is the symmedian point. Thus if AK meet the cubic again in L, the point L has the property that the hyperbola of the system that passes through it divides those hyperbolas for which the points of contact of tangents to them lie on the same branch from those for which they lie on different branches.

Again the external portions $D'B$, $D'AC$ consist of the feet of normals from A to the hyperbolas of the system, starting with the line hyperbola $A'D'$ and ending with the line hyperbola, vertices B, C. Also the portions $\infty AFDC$, ∞B consist of the feet of the normals from A to the ellipses of the system, starting with the infinite circle and ending with the line BC. The cubic is touched at T by one of these ellipses, and all smaller ellipses cut it in 6 real points, *i.e.* have four real normals from A. For the ellipse that touches the cubic two of the four normals coalesce in AT. Thus since no other ellipse and no hyperbola can be drawn to touch the cubic except at A itself, as an

examination of the figure proves, we have a theorem that only one conic of a confocal system has a point on it the centre of curvature at which is a given point A.

The figure has been drawn for the case in which B and C are acute angles. In the case of one of them being an obtuse angle, there is only a slight modification of the figure. For instance the conic in this case which has A for the centre of curvature of a point T on it is seen to be a hyperbola.

The following table follows from an examination of the figure drawn for the case of B and C acute angles:

	Number of real Tangents from A.	Number of real Normals from A.
Ellipses from circle at infinity to ellipse through A,	0	2
Ellipses from ellipse through A to ellipse through T,	2	2
Ellipses from ellipse through T to line ellipse,	2	4
Hyperbolas from line $A'D'$ to hyperbola through A,	0	2
All remaining hyperbolas,	2	2

H. L. TRACHTENBERG.

THE PSEUDO-DEFINITION OF THE STRAIGHT LINE.

WE recognise as decadence the substitution for Euclid's theorem I. 20, "Any two sides of a triangle are together greater than the third side," of the objectionable pseudo-axiom, "A straight line is the shortest line from one point to another."

This phrase, as Hilbert has well said, is of necessity meaningless when the concept "length of a curve" has not been defined. Every book on elementary geometry which has introduced as axiom or definition of a straight line any phrase equivalent to "A straight line is the shortest distance between two points," every book which has at its beginning as axiom or definition of a straight line any phrase equivalent to "A straight line is the shortest distance between two points," has been in error. We believe the advances in regard to the foundations of elementary geometry make timely the rejection of this piece of decay from the body of the elements, and that the time is ripe for the return at this point to the pristine purity of Euclid.

If any additional argument were necessary in reference to a matter so palpable, we have it in the fact that the French themselves have repudiated this pseudo-definition introduced so

unfortunately by Legendre. Of it Laisant says: "This definition, almost unanimously abandoned, represents one of the most remarkable examples of the persistence with which an absurdity can propagate itself throughout the centuries."

"In the first place, the idea expressed is incomprehensible to beginners, since it presupposes the notion of the length of a curve; and further, it is a vicious circle, since the length of a curve can only be understood as limit of a sum of rectilinear lengths; moreover, it is not a definition at all, since, on the contrary, it is a proposition or a theorem."

Reflection shows even a boy how far such a phrase is from being simple. It presupposes all the complex of assumptions necessary for measurement. But more and worse; it presumes beforehand the possibility of measuring all kinds of lines that can be drawn anywhere in space, else how could it be said that a certain one of these lines is the shortest distance between two points? Yet the straight line occurs in non-metrical, in descriptive, in pure projective geometry. All postulates demanded by metrical geometry are presupposed in this pseudo-definition. But worse; before measurement is possible we must have a standard, a unit for measurement, and the standard for length is and has always been itself the straight line. Thus measurement of length or distance presupposes the straight line as a necessary pre-condition of its own possibility, and therefore cannot be taken as a simpler and more ultimate notion with which to define the straight line, nor as a simple assumption upon which to build up the opening demonstrations of geometry. Furthermore, the very existence of a minimum is itself an assumption, and here a gratuitous and unnecessary one. These difficulties are so serious, so obvious, it is little short of a miracle that this absurd crudity has not long ago been ejected from elementary demonstrative geometry.

The common and wide-spread crude notion of the length of a string is a physical, not a geometrical notion. By making fast one end of a string and passing the other through a hole or a ring and observing how more and more of it passes out until the string becomes stretched or straight, we have an example of those experiences or experiments which confuse the physical notion of quantity of string with the geometrical notion of length of a piece of a straight line or a sect, or a combination of sects; and this physical notion is at the basis of a common mental confusion which unconsciously thinks it applicable to the length of the semi-circle. Such a mind would square or rectify the circle by putting a chalk mark on the tire of a waggon wheel, or winding a thread around a silver dollar. Such a procedure might serve to correct Solomon's value for π, but all the measurements since the world began are not

enough to prove the theorem: "Two sides of a triangle are together greater than the third."

As Dr. E. W. Hobson says: "It is a very significant fact that the operation of counting, in connection with which numbers, integral and fractional, have their origin, is the one and only absolutely exact operation of a mathematical character which we are able to undertake upon the objects which we perceive. On the other hand, all operations of the nature of measurement which we can perform in connection with the objects of perception contain an essential element of inexactness. The theory of exact measurement in the domain of the ideal objects of abstract geometry is not immediately derivable from intuition."

This assumption or definition, a straight line is the shortest of lines, can therefore have no meaning until first has been determined what geometric meaning can be given to the length of a curve.

Moreover, underneath all this is the fact that even the theorem, Euclid I. 20: "Any two sides of a triangle are together greater than the third side"; a proposition which the Sophists said even donkeys knew, simple as it is, depends upon a complex of preceding assumptions. Hilbert has thrown brilliant new light upon it in the *Proceedings of the London Mathematical Society*, 1902, pages 50-68, where he creates a geometry in which two sides of a triangle may be together less than the third side, exhibiting as a specific and definite example a right triangle in which the sum of the two sides is less than the hypothenuse.

In general the length of a curve is defined by the aggregate formed by the lengths of the perimeters of a proper sequence of inscribed polygons. This definition in ordinary cases creates, fixes for the curve a length; but in case no such aggregate is convergent, the curve is regarded as not rectifiable. Even our creative definition has failed to endow it with length, and thus lengthless it remains.

If, however, it can be shown that the lengths of the perimeters of these inscribed polygons form a convergent aggregate which is independent of the particular choice of the polygons of the sequence, the curve is rectifiable, its length being defined by the number given by the aggregate. Always the recondite idea of limit has been found necessary for defining what we are to mean by length of a curve. How visible then the blunder in trying to define the straight by a metric comparison with all curves!

What geometric meaning has the phrase: "A straight line is the shortest line that can be drawn from one point to another"? A relation of equality or inequality between two magnitudes must have some foundation, and be capable of some intelligible test. In the traditional geometry the foundation of all proof

by Euclid's method consists in establishing the congruence of magnitudes.

To make the congruence evident, the geometrical figures are supposed to be applied to one another, of course without changing their form and dimensions. But since no part of a curve can be congruent to any piece of a straight, so, for example, no part of a circle can be equivalent to any sect from the definition of equivalent magnitudes as those which can be cut into pieces congruent in pairs.

In any comparison of size by congruence, we must be able to place one of the magnitudes or portions of it in complete or partial coincidence with the other. No such direct comparison can be instituted between a straight and a line no piece of which is straight. Thus the whole of Euclid's Elements fails utterly to institute or prove any relation as regards size between a sect and an arc joining the same two points. The operation of measurement we cannot effect, rigorously speaking, either for curves or for curved surfaces, since the unit for length is a sect, and the unit for area the square on that sect. In fact, however little may be the parts of a curve, they do not cease to be curved, and consequently they cannot be compared directly with a sect; just as parts of a curved surface are not directly comparable with portions of a plane.

We cannot even affirm that any ratio exists between a circle and its diameter until after we have made some extra-Euclidean and post-Euclidean assumption.

Therefore when Phillips and Fisher, of Yale, give as their definition of a straight (1898, p. 4, No. 7, Def.): "A straight line is a line which is the shortest path between any two of its points," they pass through and beyond Euclid's Elements to give us his simplest element; they institute a metrical comparison not only with circular arcs, but also with all curves known and unknown; they presuppose a foreknowledge of all lines in a definition of the simplest line. Is it still needful to say this is grossly bad logic, bad pedagogy, bad mathematics?

There being no good reason for retaining longer this old bit of decadence, we desire the removal of this blemish, this parasite which has clung so persistently, so obstinately to what Sir Henry Savile called "the beauteous body of geometry."

GEORGE BRUCE HALSTED.

REVIEWS.

PRACTICAL (OR EXPERIMENTAL) MATHEMATICS.

ONE result of the criticisms on the teaching of Mathematics that originated in Professor Perry's address to the British Association in 1901 has been a great increase in the attention paid to practical

applications in the elementary parts of the subject. The new regulations for the Army Examinations give considerable prominence to practical work, and under the name "Practical Measurements" the subject attains a position similar to that of Practical Work in Physics and Chemistry. The new "Leaving Certificate" Examination of the Combined Universities of Oxford and Cambridge first held in July, 1905, contained experimental work of this kind, and at one centre, if not at all, candidates were required to answer a number of questions after having made measurements and experiments on a *penny*. The value of a familiar acquaintance with actual models has long been recognised, though many teachers of mathematics used nothing more than pen and paper, or chalk and blackboard. The writer, in his three years of preparation at Cambridge for the Mathematical Tripos, remembers only one practical experiment made by a lecturer; this was the rough drawing in ink of a catenary on paper; while the ink was wet, the lecturer (a famous coach of St. John's College) rolled a chalk crayon over the figure. The experiment was to illustrate a catenary on the curved surface of a cylinder. The late R. W. H. T. Hudson was an adept in practical illustrations of mathematics. While writing his remarkable book on Kummer's Quartic Surface he employed models of his own designing, constructed out of wires, etc. In his Lecture-Room at Liverpool University he had begun to collect most interesting working models, by means of which he illustrated his lectures.

The new requirements have already produced a crop of text books.

Introductory Mathematics. R. B. MORGAN. (Blackie.)

At the beginning of his preface the author states that he has attempted "to include in one cover all the Algebra, Geometry, and Graphs a boy ought to know before he can really be said to have done more than obtain an introduction to Mathematics." What he means is clear, but somewhat strangely expressed. He states further that "he has refrained from making statements which a boy ought to discover for himself."

This looseness of expression is unfortunately somewhat common in the book, and there is a slight tendency towards exaggerated language, for instance: "It is extremely important that you should know accurately"—"Accuracy is of vital importance." There are, however, many excellent features. Elementary algebraic processes are taught by appealing to existing knowledge of arithmetic, and in this connection the rules for signs are treated admirably.

The portion on equations is good, and the author never fails to verify all solutions. There is, however, an unfortunate use of the word *proof*. Thus on p. 62: "To *prove* an identity you substitute any value you please for the same letter on both sides of the identity, simplify each side separately, and show the results to be the same."

The graphical work is very fair, but the numbers giving the scales used are not always marked on the axes. The first example based on cricket scores is not well chosen, as the number of runs is not a continuous quantity.

The geometry is not quite so well done. Triangles are not touched until after the middle of the book. In classifying quadrilaterals the author omits the kite. The Church Tower on p. 107, is an unsatisfactory example, owing to the difficulty of measuring along the ground to the point vertically beneath the top of the tower.

The experimental work is rather meagre; the author apparently has not had great experience in the practical difficulties connected with this work. For instance, to measure the circumference of a circle: "Place thread or string very carefully round the circ^{ce} so as to measure its length. If you have any difficulty in preventing the thread from slipping, use pins as you proceed." Models of solids are to be made from their "nets." Very little help is given by advice as to the difficult process of joining edges together, except by recommending the use of stamp-edging. In the construction of pyramids made from paper or cardboard, the nets given are unsuitable owing to the difficulties which occur at the apex. Perhaps it was wise of the author not to mention how hard it is to fit circular bases of paper to cylinders and cones.

The following occur in the book:

$$10 \text{ mm.} = 200 \text{ feet.}$$

More briefly: $1{\cdot}5'' \times 2{\cdot}3'' = 3{\cdot}45$ sq. inches.

The book is evidently the work of an experienced teacher. It is, however, difficult to see the need for it, as most of the processes may be found in books on geometry, algebra, etc., which have been recently published. Still, it may be an advantage to have all the work in a small cheap volume, such as the one above. The printing is good. No answers are given to the examples.

Practical Mathematics. A. CONSTERDINE and A. BARNES. (John Murray.)

This book is designed chiefly for industrial students, and covers elementary geometry, algebra, and arithmetic. All the work is based on the measurement of actual objects; for instance, algebra is introduced by the formula $l = \pi d$ after experimental work on the circle. The explanations, continually interlarded with little questions, will perhaps be rather bewildering to the class of student contemplated. Rules, as such, are seldom given, but suggested for discovery.

The authors take decimals before vulgar fractions, though they introduce the former by tenths, hundredths, etc. The experimental work leading to the treatment of vulgar fractions is particularly good.

Geometrical notions are carefully introduced, but the measurement of angles in degrees might be taken earlier. It is left until nearly the middle of the book. Advice to students to make preliminary "hand-sketches" is a good feature.

The definition given of a *figure* is strange: "A plane which is bounded by one or more lines."

The treatment of fractional indices would be much improved by graphs—*e.g.* of $y = 2^x$. This method forms an excellent introduction to

logarithms, and it is a pity that the authors have not taken advantage of it.

In combining logarithms the vertical arrangement is more satisfactory. For instance the writers have

$$\log x = 2{\cdot}0913 + 1{\cdot}6586 + 0{\cdot}8318 = 4{\cdot}5817.$$

This might be written $\log x = \begin{cases} 2{\cdot}0913 \\ 1{\cdot}6586 \\ 0{\cdot}8318 \end{cases}$ or $\log x = \begin{array}{r} 2{\cdot}0913 \\ +1{\cdot}6586 \\ +0{\cdot}8318 \\ = 4{\cdot}5817 \end{array}$

The theorem of Pythagoras is suggested for discovery (1) by squared paper and the Δ 3, 4, 5; (2) by construction and calculation. One or two "dissection proofs" might well have been added. In connection with this theorem and the corresponding theorems for acute- and obtuse-angled triangles too much is expected from the student, who is, moreover, asked to discover for the first time that $(a-b)^2 = a^2 - 2ab + b^2$ from a figure corresponding to that of Euc. II. 7.

The authors speak of the "Equation of a Circle," although they state that it is very important to use the words circumference and circle in their right senses.

The chapter on Plans, Elevations, etc., is instructive, and should be of great use. It is an important branch which is often neglected in schools.

Density is not defined, but the term relative density is used for specific gravity.

A chapter is given to Trigonometry. It forms a good introduction, but perhaps more attention might have been given to the graphs of the ratios. The examples in this chapter are not well chosen, as they do not give sufficient prominence to practical applications. Moreover, no instructions are given as to the use of trigonometrical tables.

The book concludes with a short chapter on Time and Velocity.

Tables of logarithms, antilogarithms, and of values of sin, cos, and tan for angles of a whole number of degrees are placed in appendices at the end. Answers to the examples are given.

The book shows much originality, and though exception may perhaps be taken to the order and arrangement, it well fulfils the aims for which it was written.

A Note-Book on Experimental Mathematics. C. GODFREY and G. M. BELL. (Arnold.)

This book is arranged on the lines of Ashford's *Note-Book of Practical Physics*, compiled for use in Harrow School, and is intended primarily to cover the requirements of the army entrance examinations.

An list of apparatus required in the experiments is given at the beginning of the book. This should prove most useful.

There are 124 experiments with brief, but generally sufficient instructions to the student. The concrete conceptions which he should gain by working through them should be of great value to him when studying the more theoretical portions of mathematics. It is important, however, that classes taking practical work should frequently discuss their work with the teacher. It is therefore

advisable, though sometimes difficult to arrange, that the mathematical master should take both the theoretical and practical work of his form. This will generally necessitate a "Practical Mathematics Room" in addition to the Physical Laboratory. Such a room, unless the Physical Laboratory is used for the purpose, will therefore be necessary for forms in schools using this book. In fact at some of the larger schools a Practical Mathematics Room has already been established.

The book is divided into four parts; Part I. contains simple experiments and measurements connected with lengths, areas, volumes, mass and density, and fluid pressure. Part II. is on statics, with experiments on tensions, moments, and the equilibrium of three forces. The board used to illustrate the parallelogram of forces is placed in a vertical plane, and the three forces are obtained by using hanging weights. The method with three spring-balances is perhaps more easy to set up.

Part III. commences with instruments of more refined accuracy—the vernier, screw-gauge, spherometer. The vernier is introduced by a neat and convincing experiment, but the principle of a micrometer screw might well have been learnt after experiments with simple screws. The radius of a sphere is to be calculated from formula $2R=h+\frac{r^2}{h}$, r being obtained by calculating the radius of the circumcircle of the equilateral triangle formed by the three feet of the instrument. For approximate purposes a geometrical construction based on h and r (measured at once) is sufficiently accurate.

Part III. also contains further work in specific gravity and statics. It ends with a little dynamics—the determination of "g" by means of a lead ball falling against an oscillating bar, and also by the simple pendulum.

The work in this part is suggestive, and it is almost a pity that there is not more of it, *e.g.* illustrations of accelerations by means of some apparatus similar to that devised by Mr. W. C. Fletcher (referred to at the end of the book). Suggestions for further experiments are given in Part IV. The authors hint that the time has come for the retirement of Atwood's machine. It certainly deserves a rest from its old occupation of trying to discover "g."

Elementary Practical Mathematics. H. A. STERN and W. A. TOPHAM. (G. Bell & Sons.)

This contains the first nine chapters of a text book of Practical Mathematics definitely intended to satisfy the new Army Regulations. The portion given in this volume covers very well the syllabus for the "Qualifying" Examination, and its equivalent, the "Practical Measurements" of the Leaving Certificate Examination.

The first two chapters on contracted methods and graphs are scarcely necessary, as most of the work is now well done in books on arithmetic and algebra. In the graphs the writers are not uniform in indicating the scales used. It seems desirable that in *all* cases the numbers should be plainly marked on the axes. For some reason the "cooling curve" on p. 31 is inverted.

The chapters on length, mass, area, and volume are very fair. Mass and weighing are taken before areas and volumes. It is doubtful

whether this is the best plan, but the reason in this case was evidently to prepare for methods of obtaining areas and volumes by use of the balance.

Some practical work might be done with simple screws before students use the screw-gauge and spherometer, and in the application of the latter instrument for finding the radius of a sphere the authors give only the method by formula. As mentioned above the alternative method by geometrical construction drawn to scale is more easy to remember, and for approximate results is quite as good.

No reasons are given for the precautions to be taken in using a balance.

The experimental work and examples are well selected. The practical work on density is especially good. The term "specific gravity" seems a favourite one with the authors. It seems a pity that the word "density" is not used more.

Trapezoid is used for the more generally accepted trapezium, and grs. for gms.

On p. 88 occur the following statements:

loss of weight = 2·96 grs. = volume in cc.

The authors have, on the whole, provided a useful preliminary course, but perhaps more attention might be given to graphical work in connection with the experiments, *e.g.* to show experimentally that pressure varies with depth.

The diagrams are good, and the text is clear. Answers to examples are given at the end.

There is a footnote on p. 60, in which it is stated that the weight of a body at the bottom of a mine is diminished owing to the upward attraction of the mass of the earth above it. This is an unusual method of simplifying a problem in Attractions.

Practical Mathematics. STERN and TOPHAM. (Bell & Sons.)

This is the complete work of which the foregoing is a part. The additional chapters commence with the Slide Rule. This is an excellent innovation for a school text book. The authors suggest very briefly the method of making a working instrument. The actual construction is a most instructive exercise. Some excellent slide rules were made by boys at Bedford Grammar School in their spare time. A use of the slide rule which at once appeals to students is the conversion from one unit to another, *e.g.* inches to cms. and *vice versâ*. This is not explicitly mentioned in the book.

In chap. xi. the authors discuss graphs from a general point of view. For a book on Practical Mathematics it would seem more desirable to encourage students to discover general laws from the practical consideration of special cases. In fact the whole chapter is out of place, as the substance of it should be found in books on theoretical algebra.

Chaps. xii.-xiv. on vectors (principally velocities and forces), and centre of gravity, are theoretically interesting, but contain the barest amount of practical work, unless the mere construction of figures to scale is worthy of that name.

Practical work on the mechanical powers is well described, but more from the point of view of what the authors have done than of practical work to be done by the student. Work on simple levers is very meagre. The practical treatment of the wedge by apparatus of the authors' designing is, however, ingenious.

Dynamics occupies the portion of the book from chap. xvii.-xxi. A good deal of the work is theoretical (and as such, well done), but many of the experiments are beyond the range of practical work in schools.

Chap xxii. is on the Barometer and Boyle's Law. In each case the essential facts are given before any experimental work is done, the principle of the authors being apparently that the use of experiment is to verify knowledge obtained by hearsay or by reading. A training entirely according to this method seldom leads a student to original investigation, and is in many cases a bar to real progress.

Chap. xxiii. gives some interesting work (chiefly theoretical) connected with moments of inertia. Chap. xxiv. is on centres of pressure, and contains an experiment which might have been placed much earlier in the book, viz. to prove that pressure increases uniformly with depth.

The last chapter is on elasticity, torsion, etc.—subjects which more properly belong to physics.

On the whole, though the book has many good features, it is difficult to see in what way it justifies its title of "Practical Mathematics."

J. E. Boyt.

Anfangsgründe der darstellenden Geometrie für Gymnasien. By F. Schutte. (Teubner.)

A useful brochure which in the space of 41 pages explains the aim and methods of Descriptive Geometry. After a brief introduction the author proceeds to the explanation of Monge's method of representing the form of a body by means of its orthogonal projections on two planes, at right angles to each other. This occupies about half his space. His next section deals with Oblique Parallel Perspective, and the last with Central Perspective. It can be heartily recommended to those who wish to get a simple explanation of principles. We notice with interest a reference to the Stellar solids of Poinsot, one of which, the "Convexe Sterndodekaeder," is figured.

Parallelperspective—Rechtwinklige und Schiefwinklige Axonometrie. By Prof. J. Vonderlinn. (Göschen, Leipzig.) 80 pf.

An excellent work, which in 112 pages gives an account of the various methods for representing the orthogonal and oblique projections of solids, whose position is known with reference to a system of three rectangular axes. We know of no corresponding work in English. The note in the *Gazette* (177) giving the substance of De Morgan's article, "On the Projection of Mathematical Figures," will give a good idea of the nature of its contents. To any teachers who happen to be unfamiliar with this branch of mathematical study, and who wish for a simple exposition of its principles, this little book may be as heartily recommended as the preceding one. Doubtless the subject will by degrees win its way into the ordinary mathematical

curriculum, especially as the Civil Service Commissioners have introduced the elements of it into their programmes of examination work.

Planimetrie für das Gymnasium. By G. HOLZMÜLLER. Erste Teil. Zweite Auflage. (Teubner.)

A course of Plane Geometry up to and including proportion, with practical applications. It will bear comparison with any of the numerous works of the same range that have been published in the United Kingdom since 1902. The treatment is able and suggestive, the diagrams excellent and done on a generous scale, *e.g.* those on the Radical Axis and Coaxal Circles. Like Dr. Mackay, the author has inserted many historical notes and references.

Auslese aus meiner Unterrichts- und Vorlesungs-praxis. By PROF. DR. H. SCHUBERT. Erste Band. (Göschen, Leipzig.)

An interesting work dealing with several subjects, of which the ordinary mathematical student finds it difficult to obtain the information he wants without consulting large works which are difficult of access. Of its 10 sections, II. (on the division of the circle into 17 equal parts), VIII. (on the 'Konstantenzahl' of a polyhedron), and X. (Circles and Spheres), strike us as the most attractive. The last gives among its six sub-sections Steiner's solution of the problem, *To describe a circle to touch* 3 *given circles*. In I. the author shows how the logarithms of the prime numbers may be calculated successively by means of the conditions:

$$-\log(x-1)+2\log x-\log(x+1)>0,$$
$$-\log(x-1)+3\log x-3\log(x+1)+\log(x+2)>0,$$

and so on, used in the same manner as equations to eliminate all but the one sought.

Plane Geometry. Books IV. and V. By J. S. MACKAY. (W. and R. Chambers.) 2s.

An excellent little treatise giving the essentials of Euclid V. and VI., and additional theorems important in modern geometry. Practical exercises are also given. A short section on Incommensurables, which follows their treatment by Prof. G. A. Gibson, affords an introduction to their difficulties and a sufficient illustration of the way in which these difficulties may be overcome. The selection of theorems, both for book-work and exercise, seems judicious. Among these we may single out for special mention the theorems of (1) Menelaus, (2) Ceva, and (3) Desargues. Some neat arrangements are given for quickly writing down the results of (1) as referring to a set of four lines, and of (2) as referring to a set of four points. Appended to the ordinary proof of (3) is an agreeable innovation in the shape of a reference to Solid Geometry. We hope that teachers will follow the course indicated and have made by their pupils the simple models described. We think the example here set by Dr. Mackay might be profitably imitated. We are convinced that the minds of many students are stunted by being kept too long and too exclusively on plane geometry, and that an occasional excursion into three dimensional territory

would prove in many cases a useful stimulus. INVERSION is taken *pari passu* with SIMILARITY, as in the author's paper published in the *Proceedings of the Edinburgh Mathematical Society* (vol. vi. 1887-8). Interesting as this parallelism is, we are inclined to think that in working it out the author allows more space to Inversion than it deserves in a school book. We would rather see the pages occupied by applications of the theory of Similarity to geometrical problems. As the subject of *Plane Perspective* is introduced, the origin of some of the thumb rules of the draughtsman would have afforded excellent material in this connexion. Interesting algebraical applications are not wanting, and frequent historical notices are supplied from the store of erudition the author has accumulated in the course of his extensive investigations.

Elementary Modern Geometry. Part I. By H. G. WILLIS, M.A. (Clarendon Press.)

In this part the author treats of Triangles and Parallels, the discussion of the axioms on which the theory of the latter is founded being more thorough than is usual in school text-books. In the treatment favoured by the author, parallel straight lines are defined as "straight lines which meet at infinity," the meaning of the phrase having been explained by a consideration of "limiting positions," an alternative treatment by Playfair's axiom being supplied for those who are not yet advanced enough for the other. It is an agreeable change for the reviewer, and possibly for the teacher, to come across a lucid explanation of elliptic, parabolic and hyperbolic spaces in a school manual, but we have our doubts as to its effect on the mind of Jones tertius. Possibly his *blandi doctores* will allow a little skipping here and let him remain mentally a little longer in the parabolic world in which he has been used to dwell. Loci and problems receive adequate treatment, and the practical course contains some useful sections on graphs.

Experimental and Theoretical Geometry. By A. T. WARREN, M.A. (Clarendon Press.) 3rd edition.

The fact that a third edition of this treatise has been called for seems to show that its merits (to which we drew attention in the *Gazette* of July, 1903) have secured for it a substantial share of the favour of reforming teachers. A useful feature of this edition is the inclusion of various examination papers set under the new system, and occupying upwards of 30 pages. E. M. LANGLEY.

The Power of the Continuum. By DR. PITTARD-BULLOCK, Charlottenburg, Berlin, 1905. 42 pages. (Publisher's name not given.)

If we can set up a (1, 1) correspondence between the elements of two classes, we say that the classes have the same power (or cardinal number). Now, one of the most important discoveries of Georg Cantor was the proof (1873) that the class of positive integers 1, 2, 3 ... has the same power as the class of real algebraic numbers (which include the rationals) in the interval (0 ... 1), and has a less power than the continuum,—the class of all real numbers in (0 ... 1).

On the other hand, Cantor, by the construction of certain "well-ordered" series,* showed that the latter define powers which can be arranged in an unbroken ascending sequence. In particular, it results that there is a next greater power to the above power of the "enumerable" class 1, 2, 3, ...,—which latter power is seen, without difficulty, to be the least infinite power,—but it has never been proved that the power of the continuum is the least power but one, although many, including Cantor, are convinced that this is so. Dr. Pittard-Bullock's aim is to prove it.

After some historical reflexions and a repetition of some known theorems in his own terminology (*e.g.* "dinumerable" for "enumerable," "eutaxitic" for "well-ordered" †) Dr. Pittard-Bullock states the theorem (p 28),

"Every multitude of lower power than that of the n-dimensional continuum ($n>0$) is dinumerable."

Of course, from this theorem would follow that the continuum has the second power, and can be well-ordered, but Dr. Pittard-Bullock does not give the proof. He refers to a yet unpublished work of his on "Transcendental Numbers in Theory of Multitudes." We may remark, however, that the grounds he gives for asserting the above theorem do not seem sufficient. In his own words (pp. 26-28), his proof that the continuum has the second power "is partly based on a reflection that multitudes of lines, the lengths of which approach the lower limit zero (excl.), and which are found by imagining an infinite straight line divided *ad libitum*, though with the restriction, that two particles have no more than one point in common, is ‡ dinumerable. . ." Also, if A is any enumerable class, and T is the class of transcendental numbers (which is of the power of the continuum C):

"Cumulation of dinumerable submultitudes of T to A would result in a dinumerable multitude, addition of non-dimunerable submultitudes would result in a non-dinumerable multitude of at least the power of C. The third possibility, addition of a non-dinumerable multitude of dinumerable multitudes, would likewise result in a multitude equivalent to C, or of higher power, as the multitude P of points on a line, or of lines, L, the lengths of which approach the lower limit zero (excl.) are dinumerable. Assuming P to be non-dinumerable, the non-dinumerability of the first elements of the elements of P would follow, and thus the non-dinumerability of A enlarged by P."

The first statement is correct, at least if the "submultitudes" are finite in number or enumerable; the second assumes the theorem to be proved, for, if the latter be false, we can add a non-enumerable (of the second power) class to A without making it equivalent to C; the third is incomprehensible as it stands, but it may be remarked that, that the addition of a class of the second power, of enumerable aggre-

* When we speak of a *class*, we do not assume any particular *order* of the elements. A "series" is an ordered class.

† On p. 12 we read of the "arithmetics of ordinalia" (the laws of operation with ordinal numbers).

‡ Presumably *are* is meant. The theorem is Cantor's (*Math. Ann.* xx. 1882, p. 117).

gates, gives a class of the power of C assumes again the theorem to be proved.

The rest of the pamphlet is taken up with the proof that (p. 31) "the multitude W of types is also of the lowest power but one." This, as stated, is false; that the class of types of *enumerable* series is of the power of C is true (Bernstein), but irrelevant here; that the class of *enumerable, well-ordered* series is of the second power (which is what Dr. Pittard-Bullock really proves) has been proved in 1882 by Cantor, but only shows that C can be well-ordered if we have proved that there is no power between that of A and that of C.

It is easier to discover the faults of a proof when regard is paid to orthography and grammar. PHILIP E. B. JOURDAIN.

Einleitung in die Funktionentheorie. VON O. STOLZ und J. A. GMEINER (II. Abtheilung, pp. 243-598, Leipzig, Teubner, 1905).

The two parts of Stolz and Gmeiner's *Theoretische Arithmetik*, and the first part of the *Einleitung in die Funktionentheorie*, have been already noticed in these columns, and it is sufficient praise of the present volume to say that it is in every way as excellent as its predecessors. The 'second edition' of Prof. Stolz's well known *Vorlesungen ueber allgemeine Arithmetik* is now complete. No part of the original work has been more thoroughly revised or more judiciously enlarged than that which furnishes the subject matter of this volume.

A new chapter has been added on the Weierstrassian notion of the analytic function. After reading it one is disposed at first to regret that the authors do not go further into the developments of the theory, but it is probable that they have gone as far as it is wise to go without invoking the aid of the differential coefficient and the integral.

It is not easy to find anything to criticise adversely. The authors cannot lay claim to the conciseness of Jordan or the sprightliness of Picard: but in lucidity, thoroughness, and consistency of purpose they yield to no one, and although the book is not exactly easy reading it is never unreasonably difficult or heavy. It is enlivened with a large number of excellently chosen examples, many of them interesting and important theorems in themselves.

If there is one chapter which seems to me better than the rest it is chapter viii., "Die Kreisfunktionen für komplexe Werte des Arguments." The sections dealing with the equation

$$e^{\xi i} = \cos \xi + i \sin \xi,$$

the binomial theorem for a complex variable and exponent, and the functional character of the logarithm, are particularly good. The excursion into the theory of the Bernoullian numbers, and their use in asymptotic approximations, is very welcome. The chapter on Infinite Products contains a number of very useful theorems concerning products whose factors are functions of a variable, which have so far only appeared in memoirs, if at all. The chapters on continued fractions, it must be confessed, are a little less attractive; but continued fractions, indispensable and interesting as they are, are particularly difficult to treat in an attractive manner. G. H. HARDY.

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